UNIVERSITY OF NIS FACULTY OF MECHANICAL ENGINEERING

THE EIGHTH INTERNATIONAL CONFERENCE
TRANSPORT AND LOGISTICS

til 2021

MONTE CARLO METHOD BASED ON THE GAME "TOUR DU WINO" AND ITS APPLICATIONS

Predrag RAJKOVIĆ
Ljiljana RADOVIĆ
Milica BARAĆ

Faculty of Mechanical Engineering,
University of Niš, Serbia

Abstract

Random walks through a greed can be used as a simultanious and statisical method for the solving different problems. Here, we will expose a method based on the game "Tour du Wino" for the partial differential equations. But, we want to emphasize that it can inspirative for usage in the other scientific areas.

***Key words**: Monte Carlo, Random walks, Partial DE.*

1 INTRODUCTION

Monte Carlo methods, through simulations and the use of random numbers, give us quite good approximations of some, very difficult problems [1]. They are ideal for solving multidimensional problems which are very difficult or impossible to solve analytically [2].

In the paper Rajković and al. [3], it was used to approximate the multiple integrals which are very hard for exact or numerical computation. Examples of the application of the Monte Carlo method to solve differential equations and partial differential equations have indicated the efficiency of this stochastic method in solving deterministic problems. Monte Carlo methods are an ideal choice for problems with complicated initial or boundary conditions [4].

Examples of the Lotka-Volterra predator and prey model and the SIR model of epidemic spread have demonstrated the efficacy of Monte Carlo application to systems of differential equations [5].

Through the game "Tour du Wino", the application in solving partial differential equations is shown, more precisely the approximation of Dirichlet's problem [6]. This shows the adaptability and easy modification of the game to solve various border problems [7].

2 BOUNDARY VALUE PROBLEM FOR THE PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations (PDE) occur in many scientific fields. Some PDE cannot be solved analytically, then an approximation is done.

Let S be the square:

$$S = \{(x, y): \quad 0 \le x, y \le 1\}, \tag{1}$$

and ∂S be its boundary. We want to find the solution $u(x, y)$ of the elliptic PDE.

$$u_{xx} + u_{yy} = f(x, y), \tag{2}$$

which satisfies boundary conditions.

$$u(x, y)|_{\partial S} = g(x, y). \tag{3}$$

Suppose that we can not find the analityc solution. Then we can apply a few numerical methods based on the greed as this one presented on the Fig 1.

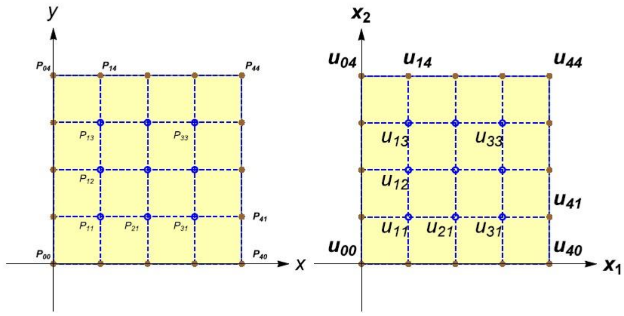

Fig. 1 *Greeds for finite differences method*

We choose partition with m rows and columns n. To make more simple, let $m = n$ and

$$h = 1/n, \quad x_i = y_i = ih \quad (i = 0,1,..., n). \tag{4}$$

Denote by

$$p_{i,j} = (x_i, y_j),\ u_{i,j} = u(p_{i,j}),\ g_{i,j} = g(p_{i,j}). \tag{5}$$

The boundary points are the points from the set

$$B = \{p_{i,0}, p_{i,n}, p_{0,i}, p_{n,i}: \quad i = 0,1,..., n\}. \tag{6}$$

Including the boundary conditions (3), we immidately have

$$u_{i,0} = g_{i,0}, \quad u_{i,n} = g_{i,n} \qquad (0 \le i \le n), \tag{7}$$

$$u_{0,j} = g_{0,j}, \quad u_{n,j} = g_{n,j} \qquad (0 \le j \le n). \tag{8}$$

The method of finite differences starts with the approximations of the second partial derivatives

$$u_{xx}(x_i, y_j) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \tag{9}$$

$$u_{yy}(x_i, y_j) \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}. \tag{10}$$

Then the PDE (2) is approximated by the system of linear algebraic equations for $i, j = 1,2..., n-1$:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f_{i,j}. \quad (11)$$

By solving this system we find $u_{i,j} \approx u(x_i, y_j)$.

But, sometimes this method requires a lot of computation which provide to the signifact loss of accurancy since the large unstable linear algebraic system should be solved.

3 MONTE CARLO METHOD FOR THE PARTIAL DIFFERENTIAL EQUATIONS

Deterministic approximation methods are very sensitive to a number of dimensions. For example, sometimes the system (8)-(9) is very hard to solve with good accuracy.
Monte Carlo methods work equally well when the problem is multidimensional. MC methods can be used to approximate many types of nonprobibalistic problems.
Due to their properties, MC methods can be very useful for approximating extremely severe PDEs. The basic idea is to solve problems of the deterministic type (problems whose outcome does not depend on chance) by methods of probibalistic type (methods whose outcome depends on chance). MC methods can be designed to approximate PDE [12], [13].
That is why we will expose a method based on the game “Tour du Wino”. It remands on a transportation problem. It is a computerized mathematical technique which simulates the behavior of a complex system, or probabilistic phenomena, using inferential statistics.

3.1. The first type

We want to find the solution $u(x,y)$ of the PDE (2) for $f(x,y) = 0,$ which satisfies boundary conditions (3).

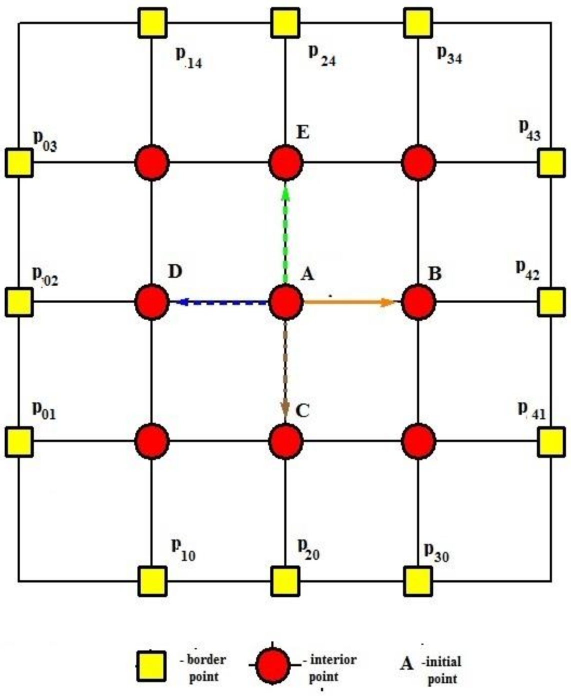

Fig. 2 *Greed for m=n=4.*

The method of finite differences uses the approximation

$$u_{i,j} \approx \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}}{4}. \quad (11)$$

Tour du Wino is played as follows:

(1) It starts from an arbitrary point (here, the point A);
(2) Denote with $W_{i,j}$ number of the cases when the wino ended at the boundary point $p_{i,j} \in B$. The initial values $W_{i,j} = 0$ are taken for every i and j;
(2) At each stage of the game, the wino goes randomly to one of the four neighboring points. The probability of going to each of these neighbors is $p = 1/4$.
(3) After arriving at a neighboring point, the wino continues this process wandering from point to point until it hits a boundary point $p_{i,j}$. Then it stops, and we increase $W_{i,j} \to W_{i,j} + 1$. This completes one random walk. See Fig. 3.
(4) We repeat steps 1-3 until the random walks $N_{\max}$ are completed. We now can compute the probability

$$P_A(p_{i,j}) = \frac{W_{i,j}}{N_{\max}}. \quad (12)$$

(5) Suppose the wino receives an award $g_{i,j}$ if it ends a walk at the boundary point $p_{i,j}$. Then the average award for all these walks is

$$R(A) = g_{0,1} P_A(p_{0,1}) + \cdots + g_{n-1,0} P_A(p_{n-1,0}). \quad (13)$$

(6) The game is finished for the point *A* by computing R(*A*). The next point should be considered.

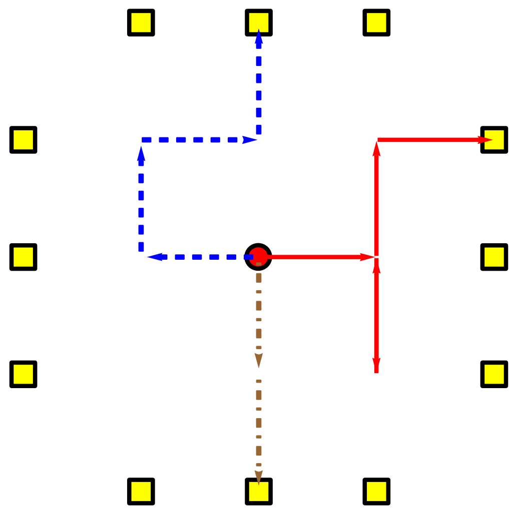

Fig. 3 *Nodes and “Tour du Wino”for m=n=4.*

It turns out that the average award is an approximation of the solution to our Dirichlet problem at point *A*. This interesting observation is based on two facts:
1. Suppose the wino started at a point *A* that was on the boundary of the square. Each resulting random walk ends immediately at that point, and the wino collects the

amount $g_{i,j}$. Thus, its average award for starting from a boundary point is also $g_{i,j}$.

2. Now suppose the wino starts from an interior point. Then, the average award *R*(*A*) is the average of the four average awards the four neighbors, i.e.

$$R(A)=\frac{R(B)+R(C)+R(D)+R(E)}{4}. \quad (13)$$

That is, *R*(*A*) corresponds to $u_{i,j}$ in the finite diference method. Hence, *R*(*A*) will approximate the true solution of the PDE at A.

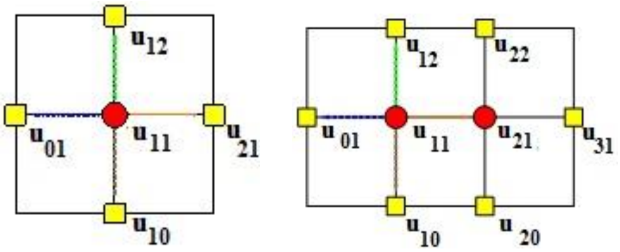

Fig. 4"*Tour de Wino" for m=n=2 and for m=2,n=3.*

Example 1. In the greed $m=2, n=3,$ from the interior point we can come to the neigbours with the same probability $1/4$. If we come to the $p_{0,1}$, $p_{1,0}$ or $p_{1,2}$ it will be the end. But, if we come to the point $p_{2,1}$, since it is not the boundary point, we should continue moving in 4 equaly valued directions. Hence

$$u_{1,1}\to\frac{1}{4}(u_{0,1}+u_{1,0}+u_{1,2}+u_{2,1}),$$

$$u_{1,1}\to\frac{1}{4}(u_{0,1}+u_{1,0}+u_{1,2})$$

$$+\frac{1}{16}(u_{1,1}+u_{2,0}+u_{3,1}+u_{2,2}),$$

$$\ldots$$

We will come to $p_{0,1}$, with the probalility:

$$u_{1,1}\to\frac{1}{4}\left(1+\frac{1}{16}+\frac{1}{16^2}+\cdots\right)=\frac{4}{15}.$$

In the same manner we can prove that:

$$u_{1,1}=\frac{4(u_{0,1}+u_{1,0}+u_{1,2})+u_{2,0}+u_{3,1}+u_{2,2}}{15},$$

$$u_{2,1}=\frac{4(u_{2,0}+u_{3,1}+u_{2,2})+u_{0,1}+u_{1,0}+u_{1,2}}{15}.$$

These formulae are the same as in the numerical solution provided by the method of finite differences.

Example 2. We want to find the solution $u(x,y)$ of the PDE (2) for $f(x,y)=0,$ which satisfies boundary conditions

$$u(0,y)=1-y,\ u(x,0)=1-x,\ u(1,y)=u(x,1)=0. \quad (14)$$

We will divide region with $n=4$ and use the experiment $N_{\max}=10\,000$ times.

Table 1 The statistics of the game "Tour de Wino" starting from the point $p_{22}=(x_2,y_2)=(0.5,0.5).$

Boundary point p_i	W_i – times the wino ends at p_i	g_i – award when the wino ends at p_i
(0.25, 0.00)	661	0.75
(0.50, 0.00)	1234	0.50
(0.75, 0.00)	650	0.25
(1.00, 0.25)	597	0
(1.00, 0.50)	1259	0
(1.00, 0.75)	613	0
(0.25, 1.00)	616	0
(0.50, 1.00)	1222	0
(0.75, 1.00)	659	0
(0.00, 0.25)	615	0.75
(0.00, 0.50)	1261	0.50
(0.00, 0.75)	613	0.25

Hence,

$$R(A)=\sum_{i=1}^{4n-4}\frac{W_i}{N_{\max}}g_i\approx 0.2502. \quad (15)$$

Continuing in that manner, we can evaluate the approximation of the solution:

$$\begin{bmatrix}u_{13} & u_{23} & u_{33}\\ u_{12} & u_{22} & u_{32}\\ u_{11} & u_{21} & u_{31}\end{bmatrix}\approx\begin{bmatrix}0.1882 & 0.1241 & 0.0645\\ 0.3731 & 0.2502 & 0.1229\\ 0.5618 & 0.3777 & 0.1885\end{bmatrix}. \quad (16)$$

The exact solution is $u(x,y)=(1-x)(1-y),$ wherefrom we find that the maximal error is $\varepsilon<0.005.$

3.2 The second type

We consider the elliptic PDE (2) on the area (1) with the boundary condtions (3).
Using the formula (10) with we find that the numerical solution is given by

$$u_{i,j}=\frac{u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}}{4}-\frac{h^2}{4}f_{i,j}. \quad (17)$$

Instead of the numerical solving the linear algebraic system we will apply the statistics of the game "|Tour de Wino":

$$R(A)=\sum_{i=1}^{4n-4}\frac{W_i}{N_{\max}}g_i-\frac{h^2}{4}f(A). \quad (18)$$

Example 3. The method based on the "Tour du Wino" can be modified for solving more general PDE's. For example, to solve the Dirichlet's problem in non-homogenuous PDE. Let S be the square (1). We want to find the solution $u(x,y)$ of the PDE

$$u_{xx}+u_{yy}=(x^2+y^2)e^{xy}, \quad (19)$$

which satisfies boundary conditions

$$u(x,0)=1, \quad u(x,1)=e^x \quad (0 \le x \le 1), \qquad (20)$$

$$u(0,y)=1, \quad u(1,y)=e^y \quad (0 \le y \le 1). \qquad (21)$$

We will use the experiment $N_{\max} = 10\,000$ times. Hence, we get the following approximation.

$$\begin{bmatrix} u_{13} & u_{23} & u_{33} \\ u_{12} & u_{22} & u_{32} \\ u_{11} & u_{21} & u_{31} \end{bmatrix} \approx \begin{bmatrix} 1.2196 & 1.4870 & 1.7768 \\ 1.1557 & 1.3311 & 1.4954 \\ 1.0806 & 1.1626 & 1.2253 \end{bmatrix}. \qquad (22)$$

In this example, the exact solution is $u(x,y)=e^{xy}$, wherefrom we find that the maximal error is $\varepsilon < 0.054$.

3.3. The third type

We want to find the solution $u(x,y)$ of the PDE:

$$u_{xx} + k\,u_{yy} = 0 \qquad (k \in N), \qquad (23)$$

which satisfies boundary condition (11).
Using the formulas (8) and (9), we find that the numerical solution is given by

$$u_{i,j} = \frac{u_{i-1,j} + u_{i+1,j} + k(u_{i,j-1} + u_{i,j+1})}{2(k+1)}. \qquad (24)$$

Instead numerical solving of the linear algebraic system we will apply the statistics of the game “|Tour de Wino“. Here, the nodes $u_{i,j-1}, u_{i,j+1}$ must have k-times more chances to be chosen then the nodes $u_{i-1,j}, u_{i+1,j}$.

Example 4. Consider:

$$u_{xx} + 4u_{yy} = 0, \qquad (27)$$

which satisfies boundary conditions

$$u(x,0) = -8x^2, \quad u(x,1) = -8x^2 + 2 \quad (0 \le x \le 1), \qquad (28)$$

$$u(0,y) = 2y^2, \quad u(1,y) = 2y^2 - 8 \quad (0 \le y \le 1). \qquad (29)$$

We will use $n=4$, the experiment $N_{\max} = 50\,000$ times and we get the following approximation

$$\begin{bmatrix} u_{13} & u_{23} & u_{33} \\ u_{12} & u_{22} & u_{32} \\ u_{11} & u_{21} & u_{31} \end{bmatrix} \approx \begin{bmatrix} 0.624 & -0.881 & -3.382 \\ 0.005 & -1.496 & -3.997 \\ -0.370 & -1.897 & -4.370 \end{bmatrix}. \qquad (30)$$

The exact solution is $u(x,y) = -8x^2 + 2y^2$, wherefrom we find that the maximal error is $\varepsilon < 0.022$.

4 CONCLUSION

In this paper, we have exposed the usage of the game “Tour du Wino“, combined with the Monte Carlo method, in the solving of the Dirichlet's problem for the elliptic partial differential equations. Monte Carlo methods, through simulations and the use of random numbers, give us quite good approximations of some, very difficult problems. They are ideal for solving multidimensional problems which are very difficult or impossible to solve analytically. This shows the adaptability and easy modification of the game to solve various border problems in other scientific areas.

ACKNOWLEDGMENT

This research was financially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

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Contact address:
Predrag Rajković
University of Niš
Faculty of Mechanical engineering
18000 Niš
A. Medvedeva 14
E-mail: predrag.rajkovic@masfak.ni.ac.rs